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LETTER TO THE EDITOR

The asymptotic form for the number of spiral self-avoiding walks

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Abstract. We investigate the asymptotic behaviour of the number s_n of spiral self-avoiding walks on the square lattice and show that $s_n \sim \rho^{\sqrt{n}}$ with $\exp(\pi/3^{1/2}) \leqslant \rho \leqslant \exp(2\pi/3^{1/2})$.

In a recent paper Privman (1983) has introduced a spiral self-avoiding walk model on the square lattice. A spiral self-avoiding walk is a self-avoiding walk with the added constraints that the walk is not allowed to make a left turn. At each step the walk must either continue in the same direction as the last step, or turn right. Privman assumes that the number s_n of such walks had the asymptotic form

$$s_n \sim n^{\gamma}\lambda^{n} \tag{1}$$

by analogy with self-avoiding walks not subject to the spiral constrant. From a series analysis study he estimated that $\lambda = 1.15 \pm 0.15$.

In this letter we present an argument, based on a connection between s_n and the number of certain types of partitions, which indicates that

$$s_n \sim \rho^{\sqrt{n}} \tag{2}$$

and we derive upper and lower bounds on the value of ρ.

Let w be a self-avoiding walk on the square lattice, with vertices numbered 0, 1, 2, ..., n having coordinates (x_k, y_k), $k = 0, 1, \ldots, n$, with $x_0 = y_0 = 0$. Since the walk is self-avoiding all vertices are distinct. If the self-avoiding walk w has no left turns then $w \in S_n$. We define the *top row* of the walk to be the set of vertices with largest x coordinate and the *top vertex* to be the vertex in the top row having largest y coordinate. Let E_n be the subset of S_n such that the top vertex of a walk in E_n has coordinates (x_n, y_n), i.e. the top vertex is also the last vertex in the walk. We write s_n and e_n for the numbers of members of S_n and E_n.

We now derive inequalities relating e_n and s_n. Clearly

$$e_n \leqslant s_n. \tag{3}$$

We define T_n to be the set of self-avoiding n-step walks with the restriction that no right turns are allowed and F_n to be the subset of T_n such that the last vertex of the walk is also the top vertex. By symmetry, T_n has s_n members and F_n has e_n members.

We now define the subset $S_n(m)$ of S_n such that $w \in S_n(m)$ if

(i) $w \in S_n$ and

(ii) the top vertex of w has coordinates (x_m, y_m), i.e. if the top vertex of w is the mth vertex of w. The $S_n(m)$ are mutually disjoint. If there are $s_n(m)$ members of

$S_n(m)$ then

$$s_n = \sum_{m=0}^{n} s_n(m). \tag{4}$$

For each $w \in E_m$ and each $w' \in F_{n-m}$, we join the two graphs by translating w' so that the graphs coincide at their top vertices. If we reverse the direction of the arrows in w' we generate a set of graphs which include all members of $S_n(m)$. Hence

$$s_n(m) \leq e_m e_{n-m}, \qquad 0 < m < n. \tag{5}$$

Clearly $s_n(n) \equiv e_n$ and $s_n(0) = e_n$. Hence

$$s_n \leq (n+1) \max_{0 \leq m \leq n} (e_m e_{n-m}) \tag{6}$$

where $e_0 = 1$.

To investigate the asymptotic behaviour of e_n we write $k_1, k_2, \ldots$ for the number of steps between successive right turns in a walk which has no left turns. This walk is a member of E_n if

$$k_1 < k_3 < k_5 < \ldots \tag{7}$$

and

$$k_2 < k_4 < k_6 < \ldots. \tag{8}$$

We can find a lower bound on e_n by looking for the number of walks which obey the more restrictive condition

$$k_1 < k_2 < k_3 \ldots. \tag{9}$$

This is just the number q_n of partitions of n into distinct integers. The generating junction of q_n is

$$Q(x) = \sum q_n x^n = (1+x)(1+x^2)(1+x^3)\ldots \tag{10}$$

and the asymptotic behaviour of q_n is

$$\log q_n \sim \pi\sqrt{n/3} \tag{11}$$

(Hardy and Ramanujan 1917). Hence

$$e_n \geq q_n \sim \exp(\pi\sqrt{n/3}). \tag{12}$$

To derive an upper bound on e_n we first note that (7) and (8) define a product of two partitions. If we write $q_{n,m}$ for the number of partitions of n into exactly m distinct integers then

$$e_n = \sum_l \sum_m (q_{l,m} q_{n-l,m} + q_{l,m} q_{n-l,m-1}) \leq \sum_l q_l q_{n-l}. \tag{13}$$

Then $e_n \leq \phi_n$ where

$$\sum_n \phi_n x^n = Q(x)^2 = \prod_{N=1}^{\infty} (1+x^N)^2. \tag{14}$$

Following Hardy and Ramanujan (1917) one can show that

$$\log \phi_n \sim 2\sqrt{cn} \tag{15}$$

where

$$c = 2\int_0^1 \frac{\log(1+t)}{t}\,dt = \pi^2/6. \tag{16}$$

Hence

$$e_n \leqslant \phi_n \sim \exp(\pi\sqrt{2n/3}). \tag{17}$$

From (3) and (12) we have

$$s_n \geqslant q_n \sim \exp(\pi\sqrt{n/3}) \tag{18}$$

and from (6) and (17)

$$s_n \leqslant (n+1) \max_{0\leqslant m\leqslant n} (\phi_m\phi_{n-m}) \sim \exp(2\pi\sqrt{n/3}). \tag{19}$$

We have been unable to prove that $\lim_{n\to\infty} n^{-1/2} \log s_n$ exists but (18) and (19) strongly suggest that

$$s_n = \rho^{\sqrt{n}+0(\sqrt{n})} \tag{20}$$

with $6.1337\ldots \leqslant \rho \leqslant 37.62\ldots$. This asymptotic behaviour is quite different from that of the number of 'unrestricted' self-avoiding walks.

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